Sample Test Group C
Grade 9 ∼ Grade 11 Part 1

1. A newly discovered Mathtium has a half-life of 7 months. How much of a 40 g sample of this new element will remain after 28 months? (Note: The half-life of a chemical substance is the time required for half of a given substance to disintegrate.)

(A) 0 **(B)** 2.1 **(C)** 2.5 **(D)** 3 **(E)** 3.5

Answer: (C)

2. What is the 2021st term in the arithmetic sequence 1, 5, 9, 13, $\cdots$?

(A) 8080 **(B)** 8081 **(C)** 8082 **(D)** 8083 **(E)** 8084

Answer: (B)

3. Consider point $P = (3, 1)$ on the Cartesian plane. If you first reflect P about the line $y = x$ and then rotate it about the origin by 90° counter-clockwise. What are the coordinates of the new position?

(A) $(1, -3)$ **(B)** $(-3, -1)$ **(C)** $(1, 3)$ **(D)** $(-3, 1)$ **(E)** $(-1, 3)$

Answer: (D)

4. Let the three roots of the equation $x^3 - 2x + 3 = 0$ be a, b and c. What is the value of

$$a^2b^2c + ab^2c^2 + a^2bc^2?$$

(A) -6 **(B)** 0 **(C)** 3 **(D)** 6 **(E)** 9

Answer: (D)

5. In Math Town, required bicycle tags consist of two letters followed by three digits. (For example, AB291, DF201, and so on.) How many distinct license tags are available?

(A) $676,000$ **(B)** $928,600$ **(C)** $1,382,400$ **(D)** $1,562,500$ **(E)** $1,982,300$

Answer: (A)

6. Determine $i \times i^3 \times i^5 \times \cdots \times i^{2021}$, where $i^2 = -1$.

(A) 1 **(B)** i **(C)** $-i$ **(D)** -1 **(E)** 0

Answer: (B)

7. What is the remainder when 3^{2021} is divided by 8?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer: (C)

8. The binary operations $*$ and $\circ$ are defined as follows:

$$x * y = x - 2y, \ x \circ y = 2x - y \text{ for all } x \text{ and } y.$$

Evaluate $(3 * 2) \circ 1$.

(A) -11 **(B)** -9 **(C)** -5 **(D)** -3 **(E)** -1

Answer: (D)

9. Let $A = (1 + \sqrt{2})^{2021}$. Determine $(1 - \sqrt{2})^{2021}$ in terms of A.

(A) $-A$ **(B)** $\frac{1}{A}$ **(C)** $1 - A$ **(D)** $-\frac{1}{A}$ **(E)** $1 + A$

Answer: (D)

10. Let a be a solution of $x^3 + 2x - 2 = 0$. Evaluate $3a^4 - a^3 + 6a^2 - 8a + 3$.

(A) 1 **(B)** 2 **(C)** $2 - \sqrt{2}$ **(D)** $2 + \sqrt{2}$ **(E)** 0

Answer: (A)

11. What is the sum of all solutions to the equation

$$|3 - |x - 2|| = 1?$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

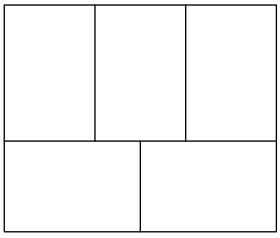
Answer: (E)

12. If the two x-intercepts and the y-intercept of $y = 24 - ax^2$ form an equilateral triangle, what is the value of a?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Answer: (A)

13. A rectangle with area 120 is divided into five congruent rectangles as shown in the diagram. What is the perimeter of one of the five congruent rectangles?



(A) 10 **(B)** 12 **(C)** 20 **(D)** 32 **(E)** 44

Answer: (C)

14. Seojin randomly distributes 9 cards with distinct integers $1, 2, 3, \cdots, 9$ into a 3×3 grid game board. What is the probability that both of the products of the numbers on the two diagonals are odd?

(A) $\frac{1}{70}$ **(B)** $\frac{1}{126}$ **(C)** $\frac{1}{128}$ **(D)** $\frac{1}{210}$ **(E)** $\frac{1}{512}$

Answer: (B)

15. How many positive integers less than 100 have exactly 8 divisors?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 10

Answer: (E)

16. Minsoo calculated two numbers, 2021_a in base a and 2021_b in base b. He found out that the difference between the two numbers is 200. What is $a + b$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Answer: (C)

17. How many positive integers less than or equal to 2021 are there in which the sum of the digits equals 5?

(A) 38 **(B)** 39 **(C)** 40 **(D)** 41 **(E)** 56

Answer: (B)

18. Suppose that a fraction $\frac{p}{q}$, where p and q are relatively prime integers, satisfies the inequality

$$\frac{19}{20} < \frac{p}{q} < \frac{20}{21}.$$

What is the smallest possible value of $p + q$?

(A) 53 **(B)** 62 **(C)** 71 **(D)** 80 **(E)** 89

Answer: (D)

19. In trapezoid $ABCD$, $\overline{AB}$ is parallel to $\overline{CD}$. Let $AB = 7$ and $CD = 10$. Two points E and F lie on $\overline{AD}$ and $\overline{BC}$, respectively. If $\overline{AB}$ is parallel to $\overline{EF}$ and $EF = 8$, what is $\frac{ED}{AE}$?

(A) 1 **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** $\frac{1}{2}$ **(E)** 2

Answer: (E)

20. Suppose that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{30} + \frac{1}{31} = \frac{p}{q},$$

where p and q are relatively prime integers. Which of the following is a divisor of p?

(A) 29 **(B)** 37 **(C)** 47 **(D)** 53 **(E)** 59

Answer: (C)

Grade 9 Part 2

1. For how many positive integers n less than 101 is $n!$ divisible by $1 + 2 + \cdots + n$?

Answer: 75

2. Let $2, 5, \cdots$ and $10, 17, \cdots$ be two arithmetic progressions. The set S is the union of the first 2021 terms of each sequence. How many distinct numbers are in S?

Answer: 3754

3. The trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 15$, $DC = 18$, and the area of $\triangle AKD$ is 30. What is the area of trapezoid $ABCD$?

Answer: 121

4. Find the sum of all the roots, real and non-real, of the equation

$$x^{2021} + \left(\frac{1}{2} - x\right)^{2021} = 0,$$

given that there are no multiple roots.

Answer: 505

5. Let $k = 2021^9 + 9^{2021}$. What is the units digit of $k^9 + 9^k$?

Answer: 1

Grade 10 Part 2

1. Let a_n be a sequence defined by $a_{n+2} = a_{n+1} \cdot a_n$ with $a_0 = 1, a_1 = 2$. What is the units digit of a_{11}?

Answer: 2

2. How many ways are there to write 8 as a sum of two or more positive integers, taking the order into account? For example, there are three ways of writing 3 as follows: $1+1+1, 2+1, 1+2$.

Answer: 127

3. How many quadratic polynomials $f(x)$ are there satisfying the condition

$$f(x^2) = f(x)f(-x)?$$

Answer: 4

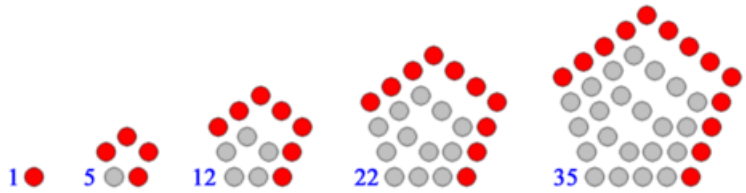
4. There are three positive real roots and one negative real root of the polynomial equation

$$(z^2 - 5z + 1)^2 - 5(z^2 - 5z + 1) + 1 = z.$$

If the negative root is written as $a - \sqrt{b}$, find the value of $a + b$.

Answer: 9

5. The first five terms of pentagonal numbers P_n are $1, 5, 12, 22, 35$ as shown in the figure.



Find the smallest positive integer $n > 1$ for which n-th pentagonal number P_n is a perfect square.

Answer: 81

Grade 11 Part 2

1. For every pair of real numbers x and y, the function f satisfies

$$xf(y) = yf(x).$$

If $f(43) = 2021$, find the value of $f(1)$.

Answer: 47

2. Suppose that α, β, and γ are three angles of a triangle. If $\tan\alpha$ and $\tan\beta$ are the roots of the equation

$$x^2 - 2021x - 2020 = 0,$$

find the angle γ in degree.

Answer: 135

3. Assume that both a and b are positive integers greater than 1. Given that the expression

$$\log_a b^4 + \log_b a^9$$

attains its minimum value, find the smallest possible value of ab.

Answer: 32

4. The Lucas sequence L_n is defined by

$$L_n = L_{n-1} + L_{n-2};\ L_0 = 2, L_1 = 1.$$

Find the value of the infinite sum

$$\sum_{n=0}^{\infty} \frac{L_n}{2^n}.$$

Answer: 6

5. Given that $z = \cos\dfrac{3\pi}{5} + i\sin\dfrac{3\pi}{5}$, evaluate

$$(3+z)(3+z^3)(3+z^5)(3+z^7)(3+z^9).$$

Answer: 242